

First-Price and Dutch

Theorem

*First-Price and Dutch auctions are **strategically equivalent**.*

- In both first-price and Dutch, a bidder must decide on the amount he's willing to pay, conditional on having placed the highest bid.
 - despite the fact that Dutch auctions are extensive-form games, the only thing a winning bidder knows about the others is that all of them have decided on lower bids
 - e.g., he does not know *what* these bids are
 - this is exactly the thing that a bidder in a first-price auction assumes when placing his bid anyway.
- Note that this is a stronger result than the connection between second-price and English.

Discussion

- So, why are both auction types held in practice?
 - First-price auctions can be held **asynchronously**
 - Dutch auctions are fast, and require **minimal communication**: only one bit needs to be transmitted from the bidders to the auctioneer.
- How should bidders bid in these auctions?

Discussion

- So, why are both auction types held in practice?
 - First-price auctions can be held **asynchronously**
 - Dutch auctions are fast, and require **minimal communication**: only one bit needs to be transmitted from the bidders to the auctioneer.
- How should bidders bid in these auctions?
 - They should clearly bid **less than their valuations**.
 - There's a tradeoff between:
 - probability of winning
 - amount paid upon winning
 - Bidders don't have a dominant strategy any more.

Analysis

Theorem

In a first-price auction with two risk-neutral bidders whose valuations are drawn independently and uniformly at random from $[0, 1]$, $(\frac{1}{2}v_1, \frac{1}{2}v_2)$ is a Bayes-Nash equilibrium strategy profile.

Proof.

Assume that bidder 2 bids $\frac{1}{2}v_2$, and bidder 1 bids s_1 . From the fact that v_2 was drawn from a uniform distribution, all values of v_2 between 0 and 1 are equally likely. Bidder 1's expected utility is

$$E[u_1] = \int_0^1 u_1 dv_2. \quad (1)$$

Note that the integral in Equation (1) can be broken up into two smaller integrals that differ on whether or not player 1 wins the auction.

$$E[u_1] = \int_0^{2s_1} u_1 dv_2 + \int_{2s_1}^1 u_1 dv_2$$

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Proof (continued).

We can now substitute in values for u_1 . In the first case, because 2 bids $\frac{1}{2}v_2$, 1 wins when $v_2 < 2s_1$, and gains utility $v_1 - s_1$. In the second case 1 loses and gains utility 0. Observe that we can ignore the case where the agents have the same valuation, because this occurs with probability zero.

$$\begin{aligned}
 E[u_1] &= \int_0^{2s_1} (v_1 - s_1) dv_2 + \int_{2s_1}^1 (0) dv_2 \\
 &= (v_1 - s_1) v_2 \Big|_0^{2s_1} \\
 &= 2v_1 s_1 - 2s_1^2
 \end{aligned} \tag{2}$$

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Proof (continued).

We can find bidder 1's best response to bidder 2's strategy by taking the derivative of Equation (2) and setting it equal to zero:

$$\begin{aligned}\frac{\partial}{\partial s_1}(2v_1s_1 - 2s_1^2) &= 0 \\ 2v_1 - 4s_1 &= 0 \\ s_1 &= \frac{1}{2}v_1\end{aligned}$$

Thus when player 2 is bidding half her valuation, player 1's best strategy is to bid half his valuation. The calculation of the optimal bid for player 2 is analogous, given the symmetry of the game and the equilibrium.

More than two bidders

- Very narrow result: two bidders, uniform valuations.
- Still, first-price auctions are not incentive compatible
 - hence, unsurprisingly, not equivalent to second-price auctions

Theorem

In a first-price sealed bid auction with n risk-neutral agents whose valuations are independently drawn from a uniform distribution on the same bounded interval of the real numbers, the unique symmetric equilibrium is given by the strategy profile

$$\left(\frac{n-1}{n}v_1, \dots, \frac{n-1}{n}v_n\right).$$

- proven using a similar argument, but more involved calculus
- a broader problem: that proof only showed how to *verify* an equilibrium strategy.
 - How do we identify one in the first place?