Revenue Equivalence

- Which auction should an auctioneer choose? To some extent, it doesn't matter...

Theorem (Revenue Equivalence Theorem)

Assume that each of n risk-neutral agents has an independent private valuation for a single good at auction, drawn from a common cumulative distribution $F(v)$ that is strictly increasing and atomless on $[\underline{v}, \bar{v}]$. Then any auction mechanism in which

- *the good will be allocated to the agent with the highest valuation; and*
- *any agent with valuation $\underline{v}$ has an expected utility of zero;*

yields the same expected revenue, and hence results in any bidder with valuation v making the same expected payment.

Revenue Equivalence Proof

Proof.

Consider any mechanism (direct or indirect) for allocating the good. Let $u_i(v_i)$ be i's expected utility given true valuation v_i, assuming that all agents including i follow their equilibrium strategies. Let $P_i(v_i)$ be i's probability of being awarded the good given (a) that his true type is v_i; (b) that he follows the equilibrium strategy for an agent with type v_i; and (c) that all other agents follow their equilibrium strategies.

$$u_i(v_i) = v_i P_i(v_i) - E[\text{payment by type } v_i \text{ of player } i] \tag{1}$$

From the definition of equilibrium, for any other valuation $\hat{v}_i$ that i could have,

$$u_i(v_i) \geq u_i(\hat{v}_i) + (v_i - \hat{v}_i) P_i(\hat{v}_i). \tag{2}$$

To understand Equation (2), observe that if i followed the equilibrium strategy for a player with valuation $\hat{v}_i$ rather than for a player with his (true) valuation v_i, i would make all the same payments and would win the good with the same probability as an agent with valuation $\hat{v}_i$. However, whenever he wins the good, i values it $(v_i - \hat{v}_i)$ more than an agent of type $\hat{v}_i$ does. The inequality must hold because in equilibrium this deviation must be unprofitable.

Revenue Equivalence Proof

Proof (continued).

Consider $\hat{v}_i = v_i + dv_i$, by substituting this expression into Equation (2):

$$u_i(v_i) \geq u_i(v_i + dv_i) + dv_i P_i(v_i + dv_i). \tag{3}$$

Likewise, considering the possibility that i's true type could be $v_i + dv_i$,

$$u_i(v_i + dv_i) \geq u_i(v_i) + dv_i P_i(v_i). \tag{4}$$

Combining Equations (4) and (5), we have

$$P_i(v_i + dv_i) \geq \frac{u_i(v_i + dv_i) - u_i(v_i)}{dv_i} \geq P_i(v_i). \tag{5}$$

Taking the limit as $dv_i \to 0$ gives $\frac{du_i}{dv_i} = P_i(v_i)$. Integrating up,

$$u_i(v_i) = u_i(\underline{v}) + \int_{x=\underline{v}}^{v_i} P_i(x)dx. \tag{6}$$

Revenue Equivalence Proof

Proof (continued).

Now consider any two efficient auction mechanisms in which the expected payment of an agent with valuation $\underline{v}$ is zero. A bidder with valuation $\underline{v}$ will never win (since the distribution is atomless), so his expected utility $u_i(\underline{v}) = 0$. Because both mechanisms are efficient, every agent i always has the same $P_i(v_i)$ (his probability of winning given his type v_i) under the two mechanisms. Since the right-hand side of Equation (6) involves only $P_i(v_i)$ and $u_i(\underline{v})$, each agent i must therefore have the same expected utility u_i in both mechanisms. From Equation (1), this means that a player of any given type v_i must make the same expected payment in both mechanisms. Thus, i's *ex ante* expected payment is also the same in both mechanisms. Since this is true for all i, the auctioneer's expected revenue is also the same in both mechanisms.

First and Second-Price Auctions

- The k^{th} order statistic of a distribution: the expected value of the k^{th}-largest of n draws.
- For n IID draws from $[0, v_{max}]$, the k^{th} order statistic is

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- First and second-price auctions satisfy the requirements of the revenue equivalence theorem
  - every symmetric game has a symmetric equilibrium
  - in a symmetric equilibrium of this auction game, higher bid $\Leftrightarrow$ higher valuation

Applying Revenue Equivalence

- Thus, a bidder in a FPA must bid his expected payment conditional on being the winner of a second-price auction
  - this conditioning will be correct if he does win the FPA; otherwise, his bid doesn't matter anyway
  - if v_i is the high value, there are then $n-1$ other values drawn from the uniform distribution on $[0, v_i]$
  - thus, the expected value of the second-highest bid is the first-order statistic of $n-1$ draws from $[0, v_i]$:

$$\frac{n+1-k}{n+1} v_{max} = \frac{(n-1)+1-(1)}{(n-1)+1}(v_i) = \frac{n-1}{n} v_i$$

- This provides a basis for our earlier claim about n-bidder first-price auctions.
  - However, we'd still have to check that this is an equilibrium
  - The revenue equivalence theorem doesn't say that every revenue-equivalent strategy profile is an equilibrium!