# Optimal Auctions

- So far we have only considered efficient auctions.
- What about maximizing the seller's revenue?
  - she may be willing to risk failing to sell the good even when there is an interested buyer
  - she may be willing sometimes to sell to a buyer who didn't make the highest bid
- Mechanisms which are designed to maximize the seller's expected revenue are known as optimal auctions.

# Optimal auctions setting

- independent private valuations
- risk-neutral bidders
- each bidder $i$'s valuation drawn from some strictly increasing cumulative density function $F_i(v)$ (PDF $f_i(v)$)
  - we allow $F_i \neq F_j$: asymmetric auctions
- the seller knows each $F_i$

# Designing optimal auctions

## Definition (virtual valuation)

Bidder $i$'s virtual valuation is $\psi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$.

## Definition (bidder-specific reserve price)

Bidder $i$'s bidder-specific reserve price $r_i^*$ is the value for which $\psi_i(r_i^*) = 0$.

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## Theorem

*The optimal (single-good) auction is a sealed-bid auction in which every agent is asked to declare his valuation. The good is sold to the agent $i = \arg\max_i \psi_i(\hat{v}_i)$, as long as $v_i > r_i^*$. If the good is sold, the winning agent $i$ is charged the smallest valuation that he could have declared while still remaining the winner:* $\inf\{v_i^* : \psi_i(v_i^*) \geq 0$ *and* $\forall j \neq i,\ \psi_i(v_i^*) \geq \psi_j(\hat{v}_j)\}$.

# Analyzing optimal auctions

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- How should bidders bid?
  - it's a second-price auction with a reserve price, held in virtual valuation space.
  - neither the reserve prices nor the virtual valuation transformation depends on the agent's declaration
  - thus the proof that a second-price auction is dominant-strategy truthful applies here as well.

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  - a second-price auction with reserve price $r^*$ satisfying $r^* - \frac{1-F_i(r^*)}{f_i(r^*)} = 0$.
- What happens in the general case?
  - the virtual valuations also increase weak bidders' bids, making them more competitive.
  - low bidders can win, paying less
  - however, bidders with higher expected valuations must bid more aggressively