Analyzing coalitional games

1. Which coalition will form?
   - we'll consider cases where the answer is the grand coalition
   - makes sense for superadditive games, like all our examples
2. How should the coalition divide its payoff?
   - in order to be fair
   - in order to be stable

Terminology:

- $\psi : \mathbb{N} \times \mathbb{R}^{2^{|N|}} \mapsto \mathbb{R}^{|N|}$: payoff division given a game
  - $\psi(N, v)$ is a vector of payoffs to each agent, explaining how they divide the payoff of the grand coalition
  - $\psi_i(N, v)$ is i's payoff
- shorthand: $x \in \mathbb{R}^N$: payoffs to each agent in N, when the game is implicit.

Axiomatizing fairness: Symmetry

As we did in social choice, let us describe fairness through axioms.

- i and j are interchangeable if they always contribute the same amount to every coalition of the other agents.
    - for all S that contains neither i nor j, $v(S \cup \{i\}) = v(S \cup \{j\})$.
- The symmetry axiom states that such agents should receive the same payments.

Axiom (Symmetry)

For any v, if i and j are interchangeable then $\psi_i(N, v) = \psi_j(N, v)$.

Axiomatizing fairness: Dummy Player

- i is a dummy player if the amount that i contributes to any coalition is exactly the amount that i is able to achieve alone.
  - for all S such that $i \notin S$, $v(S \cup \{i\}) - v(S) = v(\{i\})$.
- The dummy player axiom states that dummy players should receive a payment equal to exactly the amount that they achieve on their own.

Axiom (Dummy player)

For any v, if i is a dummy player then $\psi_i(N, v) = v(\{i\})$.

Axiomatizing fairness: Additivity

- Consider two different coalitional game theory problems, defined by two different characteristic functions v_1 and v_2, involving the same set of agents.
- The additivity axiom states that if we re-model the setting as a single game in which each coalition S achieves a payoff of $v_1(S) + v_2(S)$, the agents' payments in each coalition should be the sum of the payments they would have achieved for that coalition under the two separate games.

Axiom (Additivity)

For any two v_1 and v_2, we have for any player i that $\psi_i(N, v_1 + v_2) = \psi_i(N, v_1) + \psi_i(N, v_2)$, where the game $(N, v_1 + v_2)$ is defined by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for every coalition S.

Shapley Value

Theorem

Given a coalitional game (N, v), there is a unique payoff division $x(v) = \phi(N, v)$ that divides the full payoff of the grand coalition and that satisfies the Symmetry, Dummy player and Additivity axioms.

What is this payoff division $\phi(N, v)$? It is called the Shapley value.

Definition (Shapley value)

Given a coalitional game (N, v), the Shapley value of player i is given by

$$\phi_i(N, v) = \frac{1}{N!} \sum_{S \subseteq N \setminus \{i\}} |S|!(|N| - |S| - 1)! \Big[v(S \cup \{i\}) - v(S) \Big].$$

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This captures the “average marginal contribution” of agent i, averaging over all the different sequences according to which the grand coalition could be built up from the empty coalition.

- imagine that the coalition is assembled by starting with the empty set and adding one agent at a time, with the agent to be added chosen uniformly at random.

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- Within any such sequence of additions, look at agent i's marginal contribution at the time he is added.

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- If he is added to the set S, his contribution is $[v(S \cup \{i\}) - v(S)]$.

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- Now multiply this quantity by the $|S|!$ different ways the set S could have been formed prior to agent i's addition and by the $(|N| - |S| - 1)!$ different ways the remaining agents could be added afterward.

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- Finally, sum over all possible sets S and obtain an average by dividing by $|N|!$, the number of possible orderings of all the agents.

Shapley Value Example

Consider the Voting game:

- A, B, C, and D have 45, 25, 15, and 15 votes
- 51 votes are required to pass the $100 million bill
- A is in all winning coalitions, but doesn't win alone
- B, C, D are interchangeable: they always provide the same marginal benefit to each coalition
  - they add $100 million to the coalitions $\{B, C\}$, $\{C, D\}$, $\{B, D\}$ that do not include them already and to $\{A\}$
  - they add $0 to all other coalitions
- Grinding through the Shapley value calculation (see the book), we get the payoff division $(50, 16.66, 16.66, 16.66)$, which adds up to the entire $100 million.

Stable payoff division

- The Shapley value defined a fair way of dividing the grand coalition's payment among its members.
  - However, this analysis ignored questions of stability.
- Would the agents be willing to form the grand coalition given the way it will divide payments, or would some of them prefer to form smaller coalitions?
  - Unfortunately, sometimes smaller coalitions can be more attractive for subsets of the agents, even if they lead to lower value overall.
  - Considering the majority voting example, while A does not have a unilateral motivation to vote for a different split, A and B have incentive to defect and divide the $\$100$ million between them (e.g., $(75, 25)$).

The Core

- Under what payment divisions would the agents want to form the grand coalition?
- They would want to do so if and only if the payment profile is drawn from a set called the core.

Definition (Core)

A payoff vector x is in the core of a coalitional game (N, v) if and only if

$$\forall S \subseteq N, \sum_{i \in S} x_i \geq v(S).$$

- The sum of payoffs to the agents in any subcoalition S is at least as large as the amount that these agents could earn by forming a coalition on their own.
- Analogue to Nash equilibrium, except that it allows deviations by groups of agents.

Existence and Uniqueness

1. Is the core always nonempty?

2. Is the core always unique?

Existence and Uniqueness

1. Is the core always nonempty? No.
    - Consider again the voting game.
    - The set of minimal coalitions that meet the required 51 votes is $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, and $\{B, C, D\}$.
    - If the sum of the payoffs to parties B, C, and D is less than \$100 million, then this set of agents has incentive to deviate.
    - If B, C, and D get the entire payoff of \$100 million, then A will receive \$0 and will have incentive to form a coalition with whichever of B, C, and D obtained the smallest payoff.
    - Thus, the core is empty for this game.

2. Is the core always unique?

Existence and Uniqueness

1. Is the core always nonempty?

2. Is the core always unique? No.
    - Consider changing the example so that an 80% majority is required
    - The minimal winning coalitions are now $\{A, B, C\}$ and $\{A, B, D\}$.
    - Any complete distribution of the \$100 million among A and B now belongs to the core
        - all winning coalitions must have both the support of these two parties.

Positive results

We say that a player i is a veto player if $v(N \setminus \{i\}) = 0$.

Theorem

In a simple game the core is empty iff there is no veto player. If there are veto players, the core consists of all payoff vectors in which the nonveto players get 0*.*

Theorem

Every convex game has a nonempty core.

Theorem

In every convex game, the Shapley value is in the core.

Auction example

In our auction example, we asked whether any coalition (consisting of bidders and the seller) could do better than the payoffs they receive when everyone participates This can be rephrased as asking whether the seller's and the agents' payoffs from the auction are in the core.

Single-item, second-price auctions:

- If bidders bid truthfully, the payoffs are always in the core.
- The seller receives a revenue equal to or greater than the valuations of all the losing bidders
- Hence she cannot entice any of the losing bidders to pay her more.

VCG for combinatorial auctions

Bidder 1	Bidder 2	Bidder 3
$v_1(x,y) = 90$	$v_2(x) = v_2(x,y) = 100$	$v_3(y) = v_3(x,y) = 100$
$v_1(x) = v_1(y) = 0$	$v_2(y) = 0$	$v_3(x) = 0$

- The efficient allocation awards x to bidder 2 and y to bidder 3.
- Neither bidder is pivotal, so both pay 0.
- Both bidder 1 and the seller would benefit from forming a coalition in which bidder 1 buys the bundle x, y for any payment $0 < p_1 < 90$.
- Thus in a combinatorial auction the VCG payoffs are not guaranteed to belong to the core.