Computational Problems in Domination

- Identifying strategies dominated by a pure strategy (done)
- Identifying strategies dominated by a mixed strategy
- Identifying strategies that survive iterated elimination
- Asking whether a strategy survives iterated elimination under all elimination orderings
- We'll assume that i's utility function is strictly positive everywhere (why is this OK?)

Constraints for determining whether s_i is strictly dominated by any mixed strategy

$$\sum_{j \in A_i} p_j u_i(a_j, a_{-i}) > u_i(s_i, a_{-i}) \qquad \forall a_{-i} \in A_{-i}$$

$$p_j \geq 0 \qquad \forall j \in A_i$$

$$\sum_{j \in A_i} p_j = 1$$

Constraints for determining whether s_i is strictly dominated by any mixed strategy

$$\sum_{j \in A_i} p_j u_i(a_j, a_{-i}) > u_i(s_i, a_{-i}) \qquad \forall a_{-i} \in A_{-i}$$

$$p_j \geq 0 \qquad \forall j \in A_i$$

$$\sum_{j \in A_i} p_j = 1$$

- What's wrong with this program?

Constraints for determining whether s_i is strictly dominated by any mixed strategy

$$\sum_{j \in A_i} p_j u_i(a_j, a_{-i}) > u_i(s_i, a_{-i}) \qquad \forall a_{-i} \in A_{-i}$$

$$p_j \geq 0 \qquad \forall j \in A_i$$

$$\sum_{j \in A_i} p_j = 1$$

- What's wrong with this program?
  - strict inequality in the first constraint means we don't have an LP

LP for determining whether s_i is strictly dominated by any mixed strategy

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j \in A_i} p_j \\
\text{subject to} \quad & \sum_{j \in A_i} p_j u_i(a_j, a_{-i}) \geq u_i(s_i, a_{-i}) && \forall a_{-i} \in A_{-i} \\
& p_j \geq 0 && \forall j \in A_i
\end{aligned}
$$

- This is clearly an LP. Why is it a solution to our problem?

LP for determining whether s_i is strictly dominated by any mixed strategy

$$\begin{aligned}
\text{minimize} \quad & \sum_{j \in A_i} p_j \\
\text{subject to} \quad & \sum_{j \in A_i} p_j u_i(a_j, a_{-i}) \geq u_i(s_i, a_{-i}) && \forall a_{-i} \in A_{-i} \\
& p_j \geq 0 && \forall j \in A_i
\end{aligned}$$

- This is clearly an LP. Why is it a solution to our problem?
  - if a solution exists with $\sum_j p_j < 1$ then we can add $1 - \sum_j p_j$ to some p_k and we'll have a dominating mixed strategy (since utility was assumed to be positive everywhere)
- Our original approach works for very weak domination
- For weak domination we can use that program with a different objective function trick.

Identifying strategies that survive iterated elimination

- This can be done by repeatedly solving our LPs: solving a polynomial number of LPs is still in $\mathcal{P}$.
  - Checking whether every pure strategy of every player is dominated by any other mixed strategy requires us to solve at worst $\sum_{i \in N} |A_i|$ linear programs.
  - Each step removes one pure strategy for one player, so there can be at most $\sum_{i \in N}(|A_i| - 1)$ steps.
  - Thus we need to solve $O((n \cdot \max_i |A_i|)^2)$ linear programs.

Further questions about iterated elimination

1. **(Strategy Elimination)** Does there exist some elimination path under which the strategy s_i is eliminated?
2. **(Reduction Identity)** Given action subsets $A'_i \subseteq A_i$ for each player i, does there exist a maximally reduced game where each player i has the actions A'_i?
3. **(Uniqueness)** Does every elimination path lead to the same reduced game?
4. **(Reduction Size)** Given constants k_i for each player i, does there exist a maximally reduced game where each player i has exactly k_i actions?

Further questions about iterated elimination

1. **(Strategy Elimination)** Does there exist some elimination path under which the strategy s_i is eliminated?
2. **(Reduction Identity)** Given action subsets $A'_i \subseteq A_i$ for each player i, does there exist a maximally reduced game where each player i has the actions A'_i?
3. **(Uniqueness)** Does every elimination path lead to the same reduced game?
4. **(Reduction Size)** Given constants k_i for each player i, does there exist a maximally reduced game where each player i has exactly k_i actions?

- For iterated strict dominance these problems are all in $\mathcal{P}$.
- For iterated weak or very weak dominance these problems are all $\mathcal{NP}$-complete.