Quasilinear Utility

Definition (Quasilinear preferences)

Agents have quasilinear preferences in an n-player Bayesian game when the set of outcomes is

$$O = X \times \mathbb{R}^n$$

for a finite set X, and the utility of an agent i given joint type θ is given by

$$u_i(o, \theta) = u_i(x, \theta) - f_i(p_i),$$

where $o = (x, p)$ is an element of O, $u_i : X \times \Theta \mapsto \mathbb{R}$ is an arbitrary function and $f_i : \mathbb{R} \mapsto \mathbb{R}$ is a strictly monotonically increasing function.

Quasilinear utility

- $u_i(o, \theta) = u_i(x, \theta) - f_i(p_i)$
- We split the mechanism into a choice rule and a payment rule:
    - $x \in X$ is a discrete, non-monetary outcome
    - $p_i \in \mathbb{R}$ is a monetary payment (possibly negative) that agent i must make to the mechanism
- Implications:

Quasilinear utility

- $u_i(o, \theta) = u_i(x, \theta) - f_i(p_i)$
- We split the mechanism into a choice rule and a payment rule:
  - $x \in X$ is a discrete, non-monetary outcome
  - $p_i \in \mathbb{R}$ is a monetary payment (possibly negative) that agent i must make to the mechanism
- Implications:
  - $u_i(x, \theta)$ is not influenced by the amount of money an agent has
  - agents don't care how much others are made to pay (though they *can* care about how the choice affects others.)

Quasilinear utility

- $u_i(o, \theta) = u_i(x, \theta) - f_i(p_i)$
- We split the mechanism into a choice rule and a payment rule:
  - $x \in X$ is a discrete, non-monetary outcome
  - $p_i \in \mathbb{R}$ is a monetary payment (possibly negative) that agent i must make to the mechanism
- Implications:
  - $u_i(x, \theta)$ is not influenced by the amount of money an agent has
  - agents don't care how much others are made to pay (though they *can* care about how the choice affects others.)
- What is $f_i(p_i)$?

Risk Attitudes

- How much is \$1 worth?
  - What are the units in which this question should be answered?

Risk Attitudes

- How much is $1 worth?
  - What are the units in which this question should be answered? Utils (units of utility)

Risk Attitudes

- How much is \$1 worth?
  - What are the units in which this question should be answered? Utils (units of utility)
  - Different amounts depending on the amount of money you already have

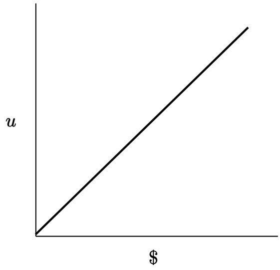
Risk Attitudes

- How much is \$1 worth?
  - What are the units in which this question should be answered? Utils (units of utility)
  - Different amounts depending on the amount of money you already have
- How much is a gamble with an expected value of \$1 worth?

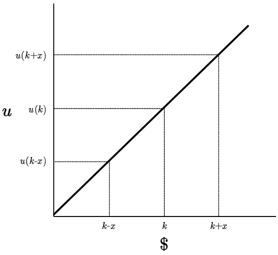
Risk Attitudes

- How much is \$1 worth?
  - What are the units in which this question should be answered? Utils (units of utility)
  - Different amounts depending on the amount of money you already have
- How much is a gamble with an expected value of \$1 worth?
  - Possibly different amounts, depending on how risky it is

Risk Neutrality

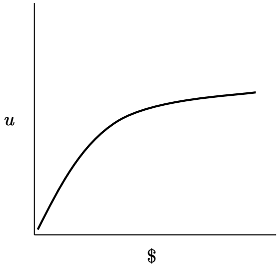


(a) Risk neutrality

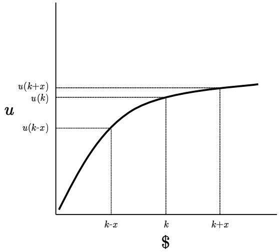


(b) Risk neutrality: fair lottery

Risk Aversion

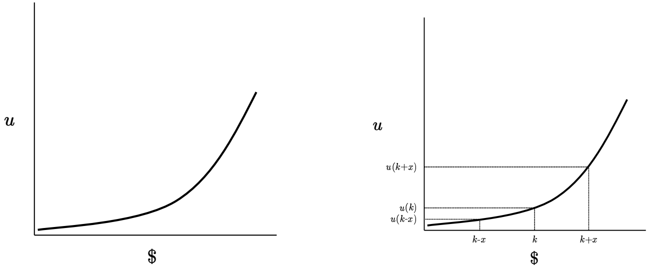


(c) Risk aversion



(d) Risk aversion: fair lottery

Risk Seeking



(e) Risk seeking

(f) Risk seeking: fair lottery

Quasilinear Mechanism

Definition (Quasilinear mechanism)

A mechanism in the quasilinear setting (for a Bayesian game setting $(N, O = X \times \mathbb{R}^n, \Theta, p, u)$) is a triple (A, χ, p), where

- $A = A_1 \times \cdots \times A_n$, where A_i is the set of actions available to agent $i \in N$,
- $\chi : A \mapsto \Pi(X)$ maps each action profile to a distribution over choices, and
- $p : A \mapsto \mathbb{R}^n$ maps each action profile to a payment for each agent.

Direct Quasilinear Mechanism

Definition (Direct quasilinear mechanism)

A direct quasilinear mechanism (for a Bayesian game setting $(N, O = X \times \mathbb{R}^n, \Theta, p, u)$) is a pair $(\chi, \wp)$. It defines a standard mechanism in the quasilinear setting, where for each i, $A_i = \Theta_i$.

Definition (Conditional utility independence)

A Bayesian game exhibits conditional utility independence if for all agents $i \in N$, for all outcomes $o \in O$ and for all pairs of joint types θ and $\theta' \in \Theta$ for which $\theta_i = \theta'_i$, it holds that $u_i(o, \theta) = u_i(o, \theta')$.

Quasilinear Mechanisms with Conditional Utility Independence

- Given conditional utility independence, we can write i's utility function as $u_i(o, \theta_i)$
  - it does not depend on the other agents' types
- An agent's valuation for choice $x \in X$: $v_i(x) = u_i(x, \theta_i)$
  - the maximum amount i would be willing to pay to get x
  - in fact, i would be indifferent between keeping the money and getting x
- Alternate definition of direct mechanism:
  - ask agents i to declare $v_i(x)$ for each $x \in X$
- Define $\hat{v}_i$ as the valuation that agent i declares to such a direct mechanism
  - may be different from his true valuation v_i
- Also define the tuples $\hat{v}$, $\hat{v}_{-i}$

Truthfulness

Definition (Truthfulness)

A quasilinear mechanism is truthful if it is direct and $\forall i \forall v_i$, agent i's equilibrium strategy is to adopt the strategy $\hat{v}_i = v_i$.

- Our definition before, adapted for the quasilinear setting

Efficiency

Definition (Efficiency)

A quasilinear mechanism is strictly Pareto efficient, or just efficient, if in equilibrium it selects a choice x such that

$$\forall v \forall x', \sum_i v_i(x) \geq \sum_i v_i(x').$$

- An efficient mechanism selects the choice that maximizes the sum of agents' utilities, disregarding monetary payments.
- How is this related to Pareto efficiency from GT?

Efficiency

Definition (Efficiency)

A quasilinear mechanism is strictly Pareto efficient, or just efficient, if in equilibrium it selects a choice x such that

$$\forall v \forall x', \sum_i v_i(x) \geq \sum_i v_i(x').$$

- An efficient mechanism selects the choice that maximizes the sum of agents' utilities, disregarding monetary payments.
- How is this related to Pareto efficiency from GT?
  - if we include the mechanism as an agent, all Pareto-efficient outcomes involve the same choice (and different payments)
  - any outcome involving another choice is Pareto-dominated: some agents could make a side-payment to others such that all would prefer the swap

Efficiency

Definition (Efficiency)

A quasilinear mechanism is strictly Pareto efficient, or just efficient, if in equilibrium it selects a choice x such that

$$\forall v \forall x', \sum_i v_i(x) \geq \sum_i v_i(x').$$

- Called economic efficiency to distinguish from other (e.g., computational) notions
- Also called social-welfare maximization
- Note: defined in terms of true (not declared) valuations.

Budget Balance

Definition (Budget balance)

A quasilinear mechanism is budget balanced when

$$\forall v, \sum_i p_i(s(v)) = 0,$$

where s is the equilibrium strategy profile.

- regardless of the agents' types, the mechanism collects and disburses the same amount of money from and to the agents

Budget Balance

Definition (Budget balance)

A quasilinear mechanism is budget balanced when

$$\forall v, \sum_i p_i(s(v)) = 0,$$

where s is the equilibrium strategy profile.

- regardless of the agents' types, the mechanism collects and disburses the same amount of money from and to the agents
- relaxed version: weak budget balance:

$$\forall v, \sum_i p_i(s(v)) \geq 0$$

  - the mechanism never takes a loss, but it may make a profit

Budget Balance

Definition (Budget balance)

A quasilinear mechanism is budget balanced when

$$\forall v, \sum_i p_i(s(v)) = 0,$$

where s is the equilibrium strategy profile.

- regardless of the agents' types, the mechanism collects and disburses the same amount of money from and to the agents
- Budget balance can be required to hold *ex ante*:

$$\mathbb{E}_v \sum_i p_i(s(v)) = 0$$

  - the mechanism must break even or make a profit only on expectation

Individual-Rationality

Definition (*Ex interim* individual rationality)

A mechanism is ex interim individual rational when $\forall i \forall v_i,\ \mathbb{E}_{v_{-i}|v_i} v_i(\chi(s_i(v_i), s_{-i}(v_{-i}))) - p_i(s_i(v_i), s_{-i}(v_{-i})) \geq 0$, where s is the equilibrium strategy profile.

- no agent loses by participating in the mechanism.
- *ex interim* because it holds for *every* possible valuation for agent i, but averages over the possible valuations of the other agents.

Individual-Rationality

Definition (*Ex interim* individual rationality)

A mechanism is ex interim individual rational when $\forall i \forall v_i,\ \mathbb{E}_{v_{-i}|v_i} v_i(\chi(s_i(v_i), s_{-i}(v_{-i}))) - p_i(s_i(v_i), s_{-i}(v_{-i})) \geq 0$, where s is the equilibrium strategy profile.

- no agent loses by participating in the mechanism.
- *ex interim* because it holds for *every* possible valuation for agent i, but averages over the possible valuations of the other agents.

Definition (*Ex post* individual rationality)

A mechanism is ex post individual rational when $\forall i \forall v,\ v_i(\chi(s(v))) - p_i(s(v)) \geq 0$, where s is the equilibrium strategy profile.

Tractability

Definition (Tractability)

A mechanism is tractable when $\forall \hat{v}$, $\chi(\hat{v})$ and $p(\hat{v})$ can be computed in polynomial time.

- The mechanism is computationally feasible.

Revenue Maximization

We can also add an objective function to our mechanism. One example: revenue maximization.

Definition (Revenue maximization)

A mechanism is revenue maximizing when, among the set of functions χ and p that satisfy the other constraints, the mechanism selects the χ and p that maximize $\mathbb{E}_\theta \sum_i p_i(s(\theta))$, where $s(\theta)$ denotes the agents' equilibrium strategy profile.

- The mechanism designer can choose among mechanisms that satisfy the desired constraints by adding an objective function such as revenue maximization.

Revenue Minimization

- The mechanism may not be intended to make money.
- Budget balance may be impossible to satisfy.
- Set weak budget balance as a constraint and add the following objective.

Definition (Revenue minimization)

A quasilinear mechanism is revenue minimizing when, among the set of functions χ and p that satisfy the other constraints, the mechanism selects the χ and p that minimize $\max_v \sum_i p_i(s(v))$ in equilibrium, where $s(v)$ denotes the agents' equilibrium strategy profile.

- Note: this considers the worst case over valuations; we could consider average case instead.

Fairness

- Fairness is hard to define. What is fairer:
  - an outcome that fines all agents \$100 and makes a choice that all agents hate equally?
  - an outcome that charges all agents \$0 and makes a choice that some agents hate and some agents like?

Fairness

- Fairness is hard to define. What is fairer:
  - an outcome that fines all agents \$100 and makes a choice that all agents hate equally?
  - an outcome that charges all agents \$0 and makes a choice that some agents hate and some agents like?
- Maxmin fairness: make the least-happy agent the happiest.

Definition (Maxmin fairness)

A quasilinear mechanism is maxmin fair when, among the set of functions χ and p that satisfy the other constraints, the mechanism selects the χ and p that maximize

$$\mathbb{E}_v \left[\min_{i \in N} v_i(\chi(s(v))) - p_i(s(v)) \right],$$

where $s(v)$ denotes the agents' equilibrium strategy profile.

Price of Anarchy Minimization

- When an efficient mechanism is impossible, we may want to get as close as possible
- Minimize the worst-case ratio between optimal social welfare and the social welfare achieved by the given mechanism.

Definition (Price-of-anarchy minimization)

A quasilinear mechanism minimizes the price of anarchy when, among the set of functions χ and p that satisfy the other constraints, the mechanism selects the χ and p that minimize

$$\max_{v \in V} \frac{\max_{x \in X} \sum_{i \in N} v_i(x)}{\sum_{i \in N} v_i\left(\chi(s(v))\right)},$$

where $s(v)$ denotes the agents' equilibrium strategy profile in the *worst* equilibrium of the mechanism—i.e., the one in which $\sum_{i \in N} v_i(\chi(s(v)))$ is the smallest.