# Two definitions

## Definition (Choice-set monotonicity)

An environment exhibits choice-set monotonicity if $\forall i, X_{-i} \subseteq X$.

- removing any agent weakly decreases—that is, never increases—the mechanism's set of possible choices $X$

## Definition (No negative externalities)

An environment exhibits no negative externalities if $\forall i \forall x \in X_{-i}, v_i(x) \geq 0$.

- every agent has zero or positive utility for any choice that can be made without his participation

# Example: road referendum

## Example

Consider the problem of holding a referendum to decide whether or not to build a road.

- The set of choices is independent of the number of agents, satisfying choice-set monotonicity.
- No agent negatively values the project, though some might value the situation in which the project is not undertaken more highly than the situation in which it is.

# Example: simple exchange

## Example

Consider a market setting consisting of agents interested in buying a single unit of a good such as a share of stock, and another set of agents interested in selling a single unit of this good. The choices in this environment are sets of buyer-seller pairings (prices are imposed through the payment function).

- If a new agent is introduced into the market, no previously-existing pairings become infeasible, but new ones become possible; thus choice-set monotonicity is satisfied.
- Because agents have zero utility both for choices that involve trades between other agents and no trades at all, there are no negative externalities.

# VCG Individual Rationality

## Theorem

*The VCG mechanism is ex-post individual rational when the choice set monotonicity and no negative externalities properties hold.*

## Proof.

All agents truthfully declare their valuations in equilibrium. Then

$$u_i = v_i(\chi(v)) - \left(\sum_{j\neq i} v_j(\chi(v_{-i})) - \sum_{j\neq i} v_j(\chi(v))\right)$$

$$= \sum_i v_i(\chi(v)) - \sum_{j\neq i} v_j(\chi(v_{-i})) \quad (1)$$

$\chi(v)$ is the outcome that maximizes social welfare, and that this optimization could have picked $\chi(v_{-i})$ instead (by choice set monotonicity). Thus,

$$\sum_j v_j(\chi(v)) \geq \sum_j v_j(\chi(v_{-i})).$$

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## Proof.

$$\sum_j v_j(\chi(v)) \geq \sum_j v_j(\chi(v_{-i})).$$

Furthermore, from no negative externalities,

$$v_i(\chi(v_{-i})) \geq 0.$$

Therefore,

$$\sum_i v_i(\chi(v)) \geq \sum_{j \neq i} v_j(\chi(v_{-i})),$$

and thus Equation (1) is non-negative.