

Analyzing Games

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- From the point of view of an outside observer, can some outcomes of a game be said to be **better** than others?
 - we have no way of saying that one agent's interests are more important than another's
 - intuition: imagine trying to find the revenue-maximizing outcome when you don't know what currency has been used to express each agent's payoff
- Are there situations where we can still prefer one outcome to another?

Pareto Optimality

- **Idea:** sometimes, one outcome o is at least as good for every agent as another outcome o' , and there is some agent who strictly prefers o to o'
 - in this case, it seems reasonable to say that o is better than o'
 - we say that o **Pareto-dominates** o' .

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 - can a game have more than one Pareto-optimal outcome?
 - does every game have at least one Pareto-optimal outcome?

Pareto Optimal Outcomes in Example Games

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- Let $a_{-i} = \langle a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \rangle$.
 - now $a = (a_{-i}, a_i)$
- **Best response:** $a_i^* \in BR(a_{-i})$ iff
$$\forall a_i \in A_i, u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$$

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- What can we say about which actions will occur?
- Idea: look for **stable** action profiles.
- $a = \langle a_1, \dots, a_n \rangle$ is a ("pure strategy") **Nash equilibrium** iff $\forall i, a_i \in BR(a_{-i})$.

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The paradox of *Prisoner's dilemma*: the Nash equilibrium is the only non-Pareto-optimal outcome!

Mixed Strategies

- It would be a pretty bad idea to play any deterministic strategy in matching pennies
- Idea: confuse the opponent by playing **randomly**
- Define a **strategy** s_i for agent i as any probability distribution over the actions A_i .
 - **pure strategy**: only one action is played with positive probability
 - **mixed strategy**: more than one action is played with positive probability
 - these actions are called the **support** of the mixed strategy
- Let the set of **all strategies** for i be S_i
- Let the set of **all strategy profiles** be $S = S_1 \times \dots \times S_n$.

Utility under Mixed Strategies

- What is your **payoff** if all the players follow mixed strategy profile $s \in S$?
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- What is your **payoff** if all the players follow mixed strategy profile $s \in S$?
 - We can't just read this number from the game matrix anymore: we won't always end up in the same cell
- Instead, use the idea of **expected utility** from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$

$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

Best Response and Nash Equilibrium

Our definitions of best response and Nash equilibrium generalize from actions to strategies.

- **Best response:**

- $s_i^* \in BR(s_{-i})$ iff $\forall s_i \in S_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$

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- **Every finite game has a Nash equilibrium!** [Nash, 1950]

- e.g., matching pennies: both players play heads/tails 50%/50%

Computing Mixed Nash Equilibria: Battle of the Sexes

	B	F
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- It's hard in general to compute Nash equilibria, but it's easy when you can guess the **support**
- For BoS, let's look for an equilibrium where all actions are part of the support

Computing Mixed Nash Equilibria: Battle of the Sexes

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- Let player 2 play B with p , F with $1 - p$.
- If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between F and B (why?)

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$$\begin{aligned}u_1(B) &= u_1(F) \\2p + 0(1 - p) &= 0p + 1(1 - p) \\p &= \frac{1}{3}\end{aligned}$$

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- Likewise, player 1 must randomize to make player 2 indifferent.
 - Why is player 1 willing to randomize?

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- Likewise, player 1 must randomize to make player 2 indifferent.
 - Why is player 1 willing to randomize?
- Let player 1 play B with q , F with $1 - q$.

$$u_2(B) = u_2(F)$$

$$q + 0(1 - q) = 0q + 2(1 - q)$$

$$q = \frac{2}{3}$$

- Thus the mixed strategies $(\frac{2}{3}, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3})$ are a Nash equilibrium.

Interpreting Mixed Strategy Equilibria

What does it mean to play a mixed strategy? Different interpretations:

- Randomize to **confuse** your opponent
 - consider the matching pennies example
- Players randomize when they are **uncertain** about the other's action
 - consider battle of the sexes
- Mixed strategies are a concise description of what might happen in **repeated play**: count of pure strategies in the limit
- Mixed strategies describe **population dynamics**: 2 agents chosen from a population, all having deterministic strategies. MS is the probability of getting an agent who will play one PS or another.